MATH 3060 Tutorial 4

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1. True or False:

   (a) If $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = |x| + |y|$, then d is a metric.
   Ans: True

   (b) $d(f, g) = \int_0^1 |f - g|^2$ is a metric on $C([0, 1])$.
   Ans: False

   (c) $d(f, g) = (\int_0^1 |f - g|^{1/2})^2$ is a metric on $C([0, 1])$.
   Ans: False

   (d) (Bolzano-Weierstrass?) Consider $X = C([-\pi, \pi])$ with the L^2 metric. Any bounded sequence of functions in X (i.e. the norms of the functions are bounded by a common constant) has a Cauchy subsequence.
   Ans: False

   (e) $(1-1 = 0?)$ Let E be a subset of a metric space X, then $X \setminus \overline{(X \setminus E)} = E^o$. $(E^{'-'} = E^o)$.
   Ans: True

   (f) Let (X, d) be a metric space. Every closed subset of X is an intersection of open subsets of X.
   Ans: True (In fact every subset is an intersection of open subsets)

   (g) Let (X, d) be a metric space. Every open subset of X is a union of closed subsets of X. Ans: True (In fact every subset is a union of closed subsets)

   (h) Let (X, d) be a metric space, and $p \in X$. Then the closure of $\{x' \in X : d(x', x) < 1\}$ in X is $\{x' \in X : d(x', x) \leq 1\}$. Ans: False

   (i) Let (X, d) be a metric space. We say a subset E of X is dense if $\overline{E} = X$. If two continuous functions $f, g : X \to \mathbb{R}$ agree on a dense subset of X, then $f = g$.
   Ans: True

   (j) There is a metric on $\mathbb{R}$, so that every subset of $\mathbb{R} \setminus \{0\}$ is open, but $\{0\}$ is not open.
   Ans: True, by the example in (a)

(k) Let (X,d) be a metric space, and suppose $X = \cup U_i$ with each U_i open. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f|_{U_i}$ is continuous for each i.
Ans: True. Take any open subset V of $\mathbb{R}$, if $f^{-1}(V) \cap U_i$ is open in U_i, then it is also open in X. But then $f^{-1}(V) = \cup_i \left(f^{-1}(V) \cap U_i\right)$.

(l) Let (X,d) be a metric space, and suppose $X = \cup F_i$ with each F_i closed. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f|_{F_i}$ is continuous for each i.
Ans: False, if we consider X to be the union of its points, then any function restricted to a point is continuous.

2. Let p be a prime number, consider the following function $N_p : \mathbb{Q} \to \mathbb{R}$. Each nonzero rational number x can be written in the form

$$x = p^n \frac{a}{b}$$

with n an integer, and a, b are integers not divisible by p. We define $N_p(x) = p^{-n}$, and also define $N_p(0) = 0$.
Show that $d(x,y) = N_p(x-y)$ is a metric on $\mathbb{Q}$. Is the sequence $1, p, p^2, p^3, \ldots$ convergent?

Proof. Let's do the triangle inequality. Let $x, y, z \in Q$. Triangle inequality trivially holds if at least two x, y, z are equal. We then assume x, y, z are distinct.
Let

$$x - y = p^n \frac{a}{b},\ y - z = p^{n'} \frac{a'}{b'}$$

with a, b, a', b' not divisible by p. Without loss of generality, assume $n \geq n'$, then

$$x - z = p^{n'} \frac{p^{n-n'} ab' + ba'}{bb'}$$

we see that

$$d(x,y) + d(y,z) - d(x,z) = p^{-n} + p^{-n'} - p^{-n'} = p^{-n} \geq 0.$$

The sequence $1, p, p^2, p^3, \ldots$ converges to 0. □

3. Let A be an $n \times n$ matrix with nonnegative entries. We say A is symmetric if $A^T = A$. We say A is disconnected if we can find a nonempty subset I of $\{1, 2, \ldots, n\}$ such that $A_{ij} = 0$ whenever $i \in I, j \in J$. We also say that A is disconnected if A is connected.

(Remark: In the tutorial, I forget the conditions that A has nonnegative entries and that I should be nonempty, they are in fact required.)

(a) If A is connected, show that for any $i, j \in \{1, 2, \dots, n\}$ there is some non negative integer k so that the (i, j) entry of A^k is nonzero. (By convention, $A^0 = I$).

(b) Assume A is symmetric and connected. For $i, j \in \{1, 2, \dots, n\}$, define $d(i, j)$ to be the minimal non negative integer k so that the (i, j) entry of A^k is nonzero. Show that d is a metric.

Proof. (a) We let $d(i, j)$ to be the infimum of the set of non negative integers k so that the (i, j) entry of A^k is nonzero. Let $i \in \{1, 2, \dots, n\}$, and

$$I = \{j \in \{1, 2, \dots, n\} : d(i, j) < \infty\}.$$

I is nonempty because $d(i, i) = 0$. We need to show that that $I = \{1, 2, \dots, n\}$. Suppose not, by the connectedness condition, we can find $j \in I, h \notin I$ and a nonnegative integer k such that the (j, h) entry of A^k is nonzero. On the other hand, since $j \in I$, we can find a nonnegative integer k' such that the (i, j) entry of $A^{k'}$ is nonzero. Combining, we see that the (i, h) entry of $A^{k'+k}$ is nonzero.

(b) Let's prove the triangle inequality. Let $i, j, h \in \{1, 2, \dots, n\}$. Suppose $d(i, j) = k, d(j, h) = k'$. Then the (i, j) entry of A^k and (j, h) entry of $A^{k'}$ are nonzero, but then the (i, h) entry of $A^{k+k'}$ is nonzero. Therefore

$$d(i, h) \le k + k' = d(i, j) + d(j, h).$$

□